stichting
mathematisch
centrum

AFDELING MATHEMATISCHE BESLISKUNDE BW 16/72 NOVEMBER

JAC.M. ANTHONISSE

THE INTERNAL STABILITY OF BIPARTITE GRAPHS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

0. Summary

A straightforward labelling procedure to obtain a maximum internally stable set of a bipartite graph from a maximum matching in that graph is described, proved correct and analysed. The number of vertices in the stable set equals the number of edges in the matching plus the number of vertices that is left exposed by the matching.

1. Introduction

Only finite, undirected graphs without loops on multiple edges will be considered. It is well-known that maximum matchings in such graphs are characterized by the absence of alternating paths connecting vertices that are left exposed by the matching. If the graph is a bipartite (or simple) one a maximum matching can be found efficiently by the Hungarian method [1].

Hakimi and Frank [2] characterized maximum internally stable sets (MIS sets) by the absence of alternating trees. They also formulated an algorithm for finding an MIS set in a bipartite graph.

In this paper another characterization of MIS sets is used. Then it is shown how, in bipartite graphs, an MIS set can be obtained from a maximum matching. The number of vertices in the MIS set equals the number of edges in the matching plus the number of vertices that is left exposed by the matching.

2. Characterization of MIS sets

V denotes the set of vertices of the graph. For a $X \subset V$, ΓX denotes the set of vertices which are adjacent to a vertex in X.

Theorem 1 (cf. [3])

S is an MIS set if and only if, for each IS set $U \subset V \setminus S$, the relation $|U| \leq |S \cap \Gamma U|$ holds.

Proof

Assume S is an MIS set and $|U| > |S \cap \Gamma U|$ for an IS set $U \subset V \setminus S$. Then the set $T = (S \setminus \Gamma U) \cup U$ is IS and $|T| > |S|$, contradicting that S is MIS. Assume both S and T are IS, with $|T| > |S|$. Let $U = T \setminus S$, then U is IS, $U \subset V \setminus S$. From $|T| = |U| + |S \cap T| > |S|$ and $S \cap \Gamma U \subset S \setminus T$ it follows that $|U| > |S| - |S \setminus T| = |S \cap T| \geq |S \cap \Gamma U|$.

Corollary 1

If S is an IS set and each vertex in $V \setminus S$ is adjacent to a different vertex in S, then S is an MIS set.

3. Labelling algorithm

Now consider a bipartite graph, thus $V = X \cup Y$ where $X \cap Y = \emptyset$ and both X and Y are IS.

A maximum matching $M = \{(x_i,y_i)\}$ $(i=1,\ldots,m)$ is assumed to be known, where (x_i,y_i) denotes an edge of the bipartite graph, $x_i \in X$, $y_i \in Y$, $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$. If $(x_i,y_i) \in M$ then x_i is the match of y_i and y_i is the match of x_i. Vertices without match are exposed vertices.

If the matching assigns a match to each vertex then $|X| = |Y|$ and, by the corollary of theorem 1, both X and Y are an MIS set.

In the labelling algorithm L below the IS set of exposed vertices is selected as part of the MIS set. Then the vertices which are adjacent to expose vertices must be excluded from the MIS set, but the matches of these vertices can be included, and so on.

Algorithm L

0) initially, all vertices are unlabelled;
1) assign label 1 to all exposed vertices, k:= 1;
2) assign label 2k to all unlabelled vertices which are adjacent to a vertex with label 2k-1, if no label can be assigned then goto 4;
3) assign label 2k+1 to the match of each vertex with label 2k, k:= k+1, return to 2;
4) assign label 2k to each unlabelled vertex in X, assign label 2k+1 to each unlabelled vertex in Y.

For the analysis of algorithm L it is convenient to introduce the following notation. At each stage of the algorithm X_i denotes the set of vertices in X with label i, Y_i denotes the set of vertices in Y with label i.

It will be shown that algorithm L assigns a label to each vertex exactly once, hence the algorithm terminates, and that the vertices with an odd label constitute an MIS set. This is obviously the case if there are no exposed vertices or if each exposed vertex is an isolated one. Thus it may be assumed now, without loss of generality, that there is at least one non-isolated exposed vertex.

Just after the first (k=1) execution of step 2 the four sets

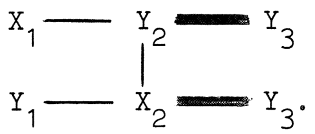
X_1 Y_2

Y_1 X_2

have been defined.

As $X_1 \cup Y_1$ = the set of exposed vertices, there is no edge between X_1 and Y_1, and no edge of M between X_1 and Y_2, nor between Y_1 and X_2. As $Y_2 = \Gamma X_1$ and $X_2 = \Gamma Y_1$ each vertex of Y_2 (resp. X_2) is adjacent to a vertex of X_1 (resp. Y_1). From the absence of alternating paths it follows that there is no edge of M between X_2 and Y_2.

The first (k=1) execution of step 3 defines X_3 and Y_3:



Each vertex in X_3 (Y_3) has its match in Y_2 (X_2), there is no edge between X_3 and Y_3. The absence of an edge of M between X_2 and Y_2 also excludes the possibility that a vertex is relabelled in this execution of step 3.

<u>Lemma 1</u>

If $x \in X_{2k}$ and $y \in Y_{2k}$ then $(x,y) \notin M$.

<u>Proof</u>

Step 3 defines X_{2k-1} as the set of matches of Y_{2k-2}, thus each vertex in X_{2k-1} has its match in Y_{2k-2} and there is no edge of M between X_{2k-1} and Y_{2k}. Each vertex in Y_{2k} is adjacent to at least one vertex in X_{2k-1}, hence each vertex in Y_{2k} is connected by an alternating path to at least one vertex in Y_{2k-2}. Consequently, each vertex in Y_{2k} is connected by an alternating path to a vertex in X_1.

Similarly, there is an alternating path from each vertex in X_{2k} to Y. Now the existence of an edge of M between X_{2k} and Y_{2k} would imply the existence of an alternating path between X_1 and Y_1, contradicting that the matching is maximum.

Lemma 2

The algorithm does not relabel a vertex.

Proof

Relabelling could occur in step 3 only, the other steps assign labels to unlabelled vertices only. Assume contrariwise that relabelling occurs, for the first time, in the k-th execution of step 3. Thus the match u of vertex v with label 2k has a label l just after the k-th execution of step 2. It is assumed that $v \in X$. If $l = 2k$ there is an edge of M between X_{2k} and Y_{2k}, contradicting lemma 1. If $l < 2k$ and l is odd, there is an edge of M between Y_l and X_{l-1} hence u has a match in both X_{2k} and X_{l-1}. If $l < 2k$ and l is even, then v was labelled already in execution l/2 of step 3. Thus in all cases a contradiction is found.

Consequently, the algorithm terminates.

Theorem 2

The vertices to which algorithm L assigns an odd label constitute an MIS set.

Proof

From the definition of algorithm L it is evident that just before the execution of step 4 no unlabelled vertex is adjacent to a vertex with an odd label. Thus if $x \in X_{2k+1}$, $y \in Y_{2l+1}$ and the edge (x,y) exists then both x and y were labelled in step 3. By the same reasoning as in the proof of lemma 1 the existence of that edge implies the existence of an alternating path between two exposed vertices.

Thus the vertices with an odd label constitute an IS set.
If $(u,v) \in M$ either u or v has an odd label. This is so because, in step 3 X_{2k+1} and Y_{2k+1} are defined by the matching, thus there is no edge of M between the vertices labelled in step 4 and the other vertices.
So each vertex with an even label has a match with an odd label and corollary 1 yields the desired result.

Corollary 2

In a bipartite graph the number of vertices in an MIS set equals the number of edges in a maximum matching plus the number of exposed vertices in such a matching.

4. Concluding remarks

The amount of computation required by steps 2 and 3 of algorithm L is proportional to, at most, the number of edges in the bipartite graph. As a maximum matching can be found by an efficient method, an MIS set can also be found efficiently.
Algorithm L is based upon the relations between maximum matchings and MIS sets as formulated in [1]. It seems difficult to exploit these relations in non-bipartite graphs. It may be of interest to note that the expression 'consider a bipartite graph' can be generalized to 'consider a graph with a given minimum coloring of the vertices'.

5. References

1. Berge, C.
The Theory of Graphs
Methuen, London, 1962.

2. Hakimi, S.L. and Frank, H.
Maximum Internally Stable Sets of a Graph
Jl. of Math. Anal. and Appl. 25 (1969) 296-308

3. Problem 230
Nieuw Archief voor Wiskunde XVII (1969) 224
(solution: XVIII (1970) 175-176)